# Similarity in Two-Dimensional Strings
# (*Extended Abstract*)

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**Abstract**

In this paper we discuss how to compute the edit distance (or similarity) between two images. We present new similarity measures and how to compute them. They can be used to perform a more general two-dimensional approximate pattern matching. Previous work on two-dimensional approximate string matching either work with only substitutions or a restricted edit distance that allows only some type of errors.

## 1 Introduction

A number of important problems related to string processing lead to algorithms for approximate string matching: text searching, pattern recognition, computational biology, audio processing, etc. Two-dimensional pattern matching with errors has applications on computer vision.

The *edit distance* between two strings $a$ and $b$, $ed(a,b)$, is defined as the minimum number of *edit operations* that must be carried out to make them equal. The allowed operations are insertion, deletion and substitution of characters in $a$ or $b$. The problem of *approximate string matching* is defined as follows: given a *text* of length $n$, and a *pattern* of length $m$, both being sequences over an alphabet $\Sigma$ of size $\sigma$, find all segments (or "occurrences") in *text* whose edit distance to *pattern* is at most $k$, where $0 < k < m$. The classical solution is $O(mn)$ time and involves dynamic programming [11].

Krithivasan and Sitalakshmi (KS) [8] proposed the following extension of edit distance for two dimensions. Given two images of the same shape, the edit distance is the sum of the edit distance of the corresponding row images. This definition is justified when the images are transmitted row by row and there are not too many communication errors. However, for many other problems, this distance does not reflect well simple cases of approximate matching in different settings. For example, we could have a match that only has the middle row of the pattern missing. In the definition above, the edit distance would be $O(m^2)$ if all pattern rows are different. Intuitively,

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the right answer should be at most $2m$, because only $m$ characters were deleted in the pattern and $m$ characters are inserted at the bottom. In this paper we extend the edit distance to two dimensions lifting the problem just mentioned and also extending the edit distance to images of different shapes.

This paper is organized as follows. First we discuss previous work on two-dimensional pattern matching with errors. Next, we introduce new notions of similarity between two-dimensional strings or images. As for the one-dimensional counterpart, we consider first the comparison of two images and then the problem of finding approximate matches with at most $k$ errors of a rectangular pattern image of size $m_1 \times m_2$ in a larger rectangular image (the text) of size $n_1 \times n_2$. We denote by $\sigma$ the size of the (finite) alphabet. We end by discussing possible extensions and open problems.

## 2 Previous Work

Two-dimensional approximate string matching usually considers only substitutions for rectangular patterns, which is much simpler than the general case with insertions and deletions (because in this case, rows and/or columns of the pattern can match pieces of the text of different length). For substitutions, the pattern shape matches the same shape in the text.

If we consider matching the pattern with at most $k$ substitutions, one of the best results on the worst case is due to Amir and Landau [2] achieving $O((k + \log \sigma)n^2)$ time but using $O(n^2)$ space. A similar algorithm is presented in Crochemore and Rytter [4]. Ranka and Heywood [10], on the other hand, solve the problem in $O((k + m)n^2)$ time and $O(kn)$ space. Amir and Landau also present a different algorithm running in $O(n^2 \log n \log\log n \log m)$ time. On average, the best algorithm is due to Karkkäinen and Ukkonen [6], with its analysis and space usage improved by Park [9]. The expected time is $O(n^2 k/m^2 \ \log_\sigma m)$ for

$$k \leq \left\lfloor \frac{m}{\lceil \log_\sigma (m^2) \rceil} \right\rfloor \frac{m}{2} - 1 \ \approx \ \frac{m^2}{4 \log_\sigma m}$$

using $O(m^2)$ space ($O(k)$ space on average). This time result is optimal for the expected case.

Under the KS definition, Krithivasan [7] presents an $O(m(k + \log m)n^2)$ algorithm that uses $O(mn)$ space. This was improved (for $k < m$) by Amir and Landau [2] to $O(k^2 n^2)$ worst case time using $O(n^2)$ space. Amir and Farach [1] also considered non-rectangular patterns achieving $O(k(k + \sqrt{m \log m}\sqrt{k \log k})n^2)$ time. This algorithm is very complicated and non-practical because it uses numerical convolutions.

Very recently, Baeza-Yates and Navarro [3] obtained the first fast algorithm on average for the KS model. They use a filter algorithm based in multiple approximate string matching, achieving $O(n^2 k \log_\sigma m \ /m^2)$ average-case behavior for $k < m(m+1)/(5 \log_\sigma m)$, and using $O(m^2)$ space. This time matches the best known result for the same problem allowing just substitutions and is optimal [6], being the upper bound on $k$ only a bit smaller. For higher error levels, they present an algorithm with time complexity $O(n^2 k/(w\sqrt{\sigma}))$ (where $w$ is the size in bits of the computer word), which works for $k < m(m+1)(1 - e/\sqrt{\sigma})$.

Another related problem is geometric matching, where we have to match a geometric figure or a set of points. In this case, the problem is in a continuous space rather than a discrete space and usually the Hausdorff measure is used.

## 3 Extending the Edit Distance

Let $a$ and $b$ be two images of size $nr \times nc$ and $mr \times mc$ respectively. In the sequel we use $row_i(a)$ to denote the $i$-th row of $a$ and $col_i(a)$ to denote the $i$-th column of $a$. For example, the KS distance is given by

$$KS(a,b) = \sum_{i=1}^{nr} ed(row_i(a), row_i(b))$$

with the restriction that $nr = mr$. We also use the L-shape idea of Giancarlo [5] used for extending suffix trees to two dimensions. We denote by $LS_{i,j}(a)$ the L-shaped string consisting of the first $j$ elements of the $i$-th row and the first $i-1$ elements of the $j$-th column.

Because our main motivation is approximate matching, we assume that the pattern and a text subimage are compared from top to bottom and from left to right. That is, the incremental computation can be decomposed by extending a sub-image at the bottom or/and the right side. It can be argued that a pattern can match better fixing a different corner, but this does not make any difference, because that only changes the text position where the match will be reported, and still only one match is found. Another convention is that the text occurrence must have the same shape of the pattern. Otherwise, we may have occurrences that have at most $k$ errors that basically do not count unmatched characters on the boundaries, which is not fair. Hence, although our similarity measures work for two images of different size, they will be used later for subimages in the text that have the same shape as the pattern.

First, we solve the limitation of the KS model to handle deletions or insertions of whole rows. The solution is simple, we just treat each row as a single string which is compared to other rows using the normal edit distance (that is, only one dimension). If $R_{i,j}$ is the distance between rows 1 and $j$ of image $a$ and rows 1 and $j$ of image $b$, we have that

$$R_{i,j} = \min(R_{i-1,j} + nc, R_{i,j-1} + mc, R_{i-1,j-1} + ed(row_i(a), row_j(b)))$$

where the boundary conditions are $R_{i,0} = i \cdot nc$ and $R_{0,j} = j \cdot mc$, and the distance between the two images is given by $R(a,b) = R_{nr,mr}$.

In the example given in the introduction, the distance is reduced to less or equal than $2m$ instead of being $O(m^2)$ as in the KS model. Similarly, we could use columns instead of rows. This model is much more fair than the KS model. Although we use rectangular images, this measure also works for any images where rows are connected and continuous.

To generalize this idea to insertions and deletions at the same time in rows and/or columns is not as simple. Suppose that we have two subimages that we want to compare. One alternative is to decompose the border of a subimage in rows or columns. Then we can extend dynamic programming by

1. removing one row or one column from one of the subimages or
2. removing one row or one column in the same side of each subimage and computing the edit distance between them.

Therefore, if $RC_{i,j,k,\ell}$ is the distance between the left-top corner of $a$ bounded by row $i$ and column $j$ and the left-top corner of $b$ bounded by row $k$ and column $\ell$, we have that $RC_{i,j,k,\ell}$ is the minimum of the following values:

- $RC_{i-1,j,k,\ell}+j$, $RC_{i,j-1,k,\ell}+i$, $RC_{i,j,k-1,\ell}+\ell$, and $RC_{i,j,k,\ell-1}+k$ which corresponds to deleting one row or column in one sub-image; and
- $RC_{i-1,j,k-1,\ell}+ed(row_i(a), row_k(b)))$ and $RC_{i,j-1,k,\ell-1}+ed(col_j(a), col_\ell(b)))$ which corresponds to comparing two rows at the bottom or two columns at the right.

The boundary conditions are $RC_{0,0,i,j} = RC_{i,j,0,0} = i \cdot j$. The distance $RC(a,b)$ is given by $R_{nr,nc,mr,mc}$. Figure 1 shows all these cases. This distance can also be applied to any convex image, for example circles or other regular polygons.

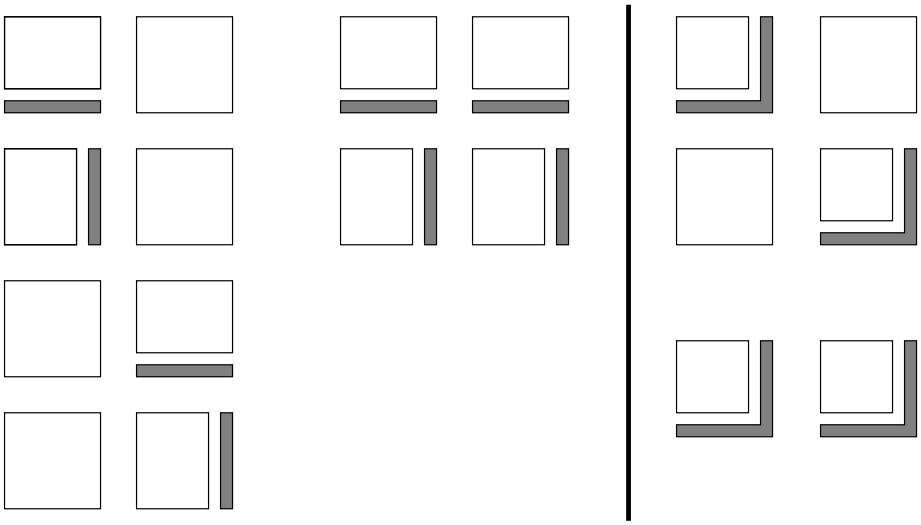

Figure 1: Decomposition used in $RC$ (left, 6 cases) and $L$ (right, 3 cases).

Nevertheless, this distance does not handle cases where we want to change at the same time a row and a column. For that we use the L-shape mentioned earlier. So, we can also decompose the border of a subimage using L-shapes and we can have the same extensions as for rows or columns. To compare two L-shapes we see them as two one-dimensional strings. Then we have the following cases to find the minimal decomposed distance:

- $L_{i-1,j-1,k,\ell} + i + j - 1$ and $L_{i,j,k-1,\ell-1} + k + \ell - 1$ which corresponds to removing an L-shape in a subimage; and
- $L_{i-1,j-1,k-1,\ell-1} + ed(LS_{i,j}(a), LS_{k,\ell}(b)))$ which corresponds to comparing two L-shapes.

The boundary conditions are the same as the $RC$ measure and the final distance is similarly given by $L(a,b) = L_{nr,nc,mr,mc}$. Figure 1 shows the decompositions associated to $L$.

Finally, we can have a general distance $All(a,b)$ that uses both decompositions at the same time ($RC$ and $L$) computing the minimal value of all possible cases. It is easy to show that $KS(a,b) \geq R(a,b) \geq RC(a,b) \geq All(a,b)$ and that $L(a,b) \geq All(a,b)$ because each case is a subset of the next. On the other hand, there are cases where $RC(a,b)$ will be less than $L(a,b)$ and vice versa. Specific examples will be included later.

## 4 Algorithmic Issues

For sake of simplicity and without loss of generality assume that $nr = nc = mr = mc = m$. A direct implementation for $R$ would take $O(m^4)$ time and $O(m^2)$ space, while for $RC$ and $L$ would require $O(m^6)$ time and $O(m^4)$ space. The later, in particular, is prohibitive even for small images. However, this can be done better. The space is easily reduced to $O(m)$ for $R$ and $O(m^2)$ to $RC$ and $L$ by noticing that we only need to store the boundary of the matrices of the dynamic programming computation as they are computed incrementally. The edit distances involved in both cases can also be computed incrementally. This needs additional space which matches the improved space bound just mentioned. Therefore, each edit distance needed in the dynamic programming computation can be computed in constant time, reducing the total time to $O(m^2)$ for $R$ and $O(m^4)$ for $RC$ and $LC$, and hence for $All$. In the full version we will include an appendix with the C code of the improved computation. This implementation allows to handle patterns up to size $50 \times 50$ without performance problems.

Now we discuss approximate two-dimensional pattern matching. For the $R$ measure we can use the same fast expected time algorithm of Baeza-Yates and Navarro [3]. This algorithm uses a filter that searches all the pattern rows with a multiple approximate string matching algorithm to find potential matching areas. We only change the verification phase (each potential match found by the filter must be verified) by using $R$ to compute the distance. Because computing $R$ takes the same time of computing $KS$, we obtain the same time bounds and the same error level bound for whic the expected time result is valid. That is, $O(n^2 k \log_\sigma m \,/m^2)$ average-case time for $k < m(m+1)/(5 \log_\sigma m)$, using $O(m^2)$ space. This expected time is optimal [6].

For the $RC$ measure, we can use the same algorithm by applying it twice. First based on the pattern rows and second on the pattern columns. Therefore we achieve the same expected time and space complexity. We are currently adapting this algorithm to handle $L$-shapes and we plan to include some preliminary results on the full version.

## 5 Concluding Remarks

Our measures can be easily extended to more dimensions. For $d$-dimensions we use $(d-1)$-dimensional strings for the decompositions. The only drawback is that the number of cases grows exponentially with the number of dimensions. Then, computing $All(a, b)$ for $d$-dimensional strings would require $O(2^d n^{2d})$ time and $O(n^d)$ space. An open problem is to design optimal worst-case time algorithms for approximate searching using the new measures. That is, achieving $O(n^4)$ time complexity in the case of two dimensions.

Neither of the new measures defined can handle scaling transformations nor rotations. A more realistic distance can be defined using the following idea, which tries to define the *largest common image* of two images, which generalizes the concept of longest common subsequence of one-dimensional strings. Given two images, find a set of position pairs that match exactly in both images subject to the following restrictions:

1. The set of positions for the same pattern are disjoint;
2. a suitable order given by the position values is the same for both images (for example, image pixels can be sorted by their $i + j$ value, using the value of $i$ in the case of ties); and

3. the total size of the set of positions is maximized.

For the edit distance, condition 3 has to be changed to:

3. Minimize the number of mismatches, insertions and deletions needed to obtain the set of matching positions.

This model may match a rotated pattern, because no corner is fixed. Figure 2 gives an example. All pieces of the pattern not in the text corresponds to deletions and mismatches and should be counted. In the text, black regions are not counted, because correspond to mismatches. All other pieces are insertions in the pattern. It is not clear that the minimal string editing solution gives the same answer as the largest common set of sub-images. Also, it could be argued that characters inserted/deleted on external borders should not be counted as errors.

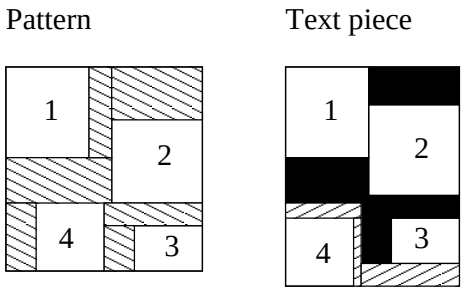


Figure 2: Example of largest common image.

The approximate two-dimensional pattern matching problem can be stated as usual using the above definition as searching for all rectangular subimages of the text that have edit distance at most $k$ with the pattern. An alternative definition would be to find all pieces of the text that have at least $m^2 - k$ matching positions with the pattern.

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