Semigroup Forum Vol. 61 (2000) 154–158

DOI: 10.1007/s002339910051

SHORT NOTE

On Free Inverse Semigroups

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Communicated by Gerard J. Lallement

Abstract

Using techniques of Rewriting Theory, we present a new proof of the known theorem of Munn that FI_X, the free inverse semigroup on X, is isomorphic to birooted word-trees on X.

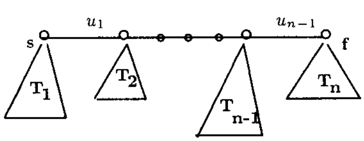
Inverse semigroups form a variety of semigroups with one additional unary operation, $(\ )^{-1}$. The following set I of equations axiomatizes it (see [1]):

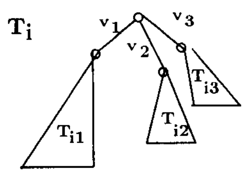
$$x(yz) = (xy)z \tag{1}$$
$$(x^{-1})^{-1} = x \tag{2}$$
$$(xy)^{-1} = y^{-1}x^{-1} \tag{3}$$
$$xx^{-1}x = x \tag{4}$$
$$xx^{-1}yy^{-1} = yy^{-1}xx^{-1}. \tag{5}$$

The set of *terms* over X, T_X, is defined recursively by: (1) Every $x \in X$ is a term, (2) If t_1, t_2 are terms, then t_1t_2 and $(t_1)^{-1}$ are terms. We will avoid superfluous parentheses, e.g. will write xyz instead of $(xy)z$ or $x(yz)$. Note that for each term $t \in T_X$, there is an I-equivalent term $w_t = u_1 \cdots u_n$, with $u_i \in X \cup \{a^{-1} : a \in X\}$ (apply repeatedly Ax. (2) and (3)).

For a given set X, denote by G_X the set of finite, directed, acyclic graphs (i.e. trees) whose edges are labeled by elements of X, with two distinguished vertices, s and f. A *birooted word-tree* on X is a tree in G_X which does not contain subgraphs of the form $\bullet \xleftarrow{a} \bullet \xrightarrow{a} \bullet$ or $\bullet \xrightarrow{a} \bullet \xleftarrow{a} \bullet$, where $a \in X$.

If $g \in G_X$ then g is a tree, hence between s and f there is exactly one simple path ${}_s\bullet \overset{u_1}{—} \bullet \overset{u_2}{—} \bullet \cdots \bullet \overset{u_{n-1}}{—} \bullet_f$ where $\bullet \overset{u_i}{—} \bullet$ represent $\bullet \xrightarrow{a} \bullet$ if $u_i = a$, and $\bullet \xleftarrow{a} \bullet$ if $u_i = a^{-1}$. So, each $g \in G_X$ has the form of the graph on the left:





and the T_i's are directed labeled trees (see the graph on the right). Hence, formally we can describe each $g \in G_X$ as:

$$g = (T_1, u_1, \ldots, T_{n-1}, u_{n-1}, T_n), \tag{6}$$

where each T_i is defined recursively as a multiset $T_i = \{v_1T_{i1}, \ldots, v_{j_i}T_{ij_i}\}$ or $T_i = \bullet$, with $u_i, v_j \in X \cup \{a^{-1} : a \in X\}$.

For $g_1, g_2 \in G_X$, define $(g_1)^{-1}$ as the same tree as g_1 but with s and f interchanged, and g_1g_2 as the tree obtained from g_1 and g_2 by identifying the vertices f_1 and s_2, and distinguishing s_1 and f_2. So, G_X becomes an algebra over the same signature as I which clearly satisfies axioms (1)-(3).

Lemma 1. *Let g be as in (6). The following functions are well defined:*

1. *$\tau : G_X \longrightarrow T_X$ defined by $\tau(g) = \tau(T_1)u_1\tau(T_2)\cdots u_{n-1}\tau(T_n)$ and $\tau(T_i) = v_1\tau(T_{i1})v_1^{-1}\cdots v_{j_i}\tau(T_{ij_i})v_{j_i}^{-1}$ and $\tau(\bullet) = \epsilon$, the empty word.*

2. *$\gamma : T_X \longrightarrow G_X$, defined by $\gamma(a) = {}_s\bullet\xrightarrow{a}\bullet_f$ for $a \in X$, and extended recursively by $\gamma(t_1t_2) = \gamma(t_1)\gamma(t_2)$ and $\gamma(t^{-1}) = \gamma(t)^{-1}$.* ■

Definition 1. Define in G_X the binary relation $\Longrightarrow$ by

$$_p\bullet\xleftarrow{a}\bullet_q\xrightarrow{a}\bullet_r \quad\Longrightarrow\quad {}_q\bullet\xrightarrow{a}\bullet_{p=r} \tag{7}$$

$$_p\bullet\xrightarrow{a}\bullet_q\xleftarrow{a}\bullet_r \quad\Longrightarrow\quad {}_q\bullet\xleftarrow{x}\bullet_{p=r} \tag{8}$$

for every $a \in X$. Read "the graph on the left rewrites to the one on the right". Extend it to all graphs in G_X by defining $g \Longrightarrow g'$ iff for some $a \in X$ the left hand side of (7) or (8) is a subgraph of g, and g' is obtained from g by identyfying the vertices p and r and eliminating the repeated edge. The symbol $\stackrel{*}{\Longrightarrow}$ denotes the reflexive-transitive closure of $\Longrightarrow$, and $\stackrel{*}{\Longleftrightarrow}$ the reflexive, symmetric and transitive closure of $\Longrightarrow$ (i.e. $g \stackrel{*}{\Longleftrightarrow} h$ if there is a sequence $g = g_1, \ldots, g_m = h$ with $g_i \Longrightarrow g_{i+1}$ or $g_i \Longleftarrow g_{i+1}$ for each $i = 1, \ldots, m-1$, $m \geq 1$).

Lemma 2. *Let $r, t \in T_X$, and let g, h be graphs in G_X.*

1. *If $I \vdash r = t$ then $\gamma(r) \stackrel{*}{\Longleftrightarrow} \gamma(t)$.*

2. *If $g \stackrel{*}{\Longleftrightarrow} h$ then $I \vdash \tau(g) = \tau(h)$.*

Proof. (1) Note that γ is an homomorphism, hence it is enough to prove the case when $r = t$ is an axiom in I. The cases of Ax. (1)-(3) are trivial. For Ax. (4) and (5) use the fact that given a term t and $w_t = u_1 \cdots u_n$, then $\gamma(t) = \gamma(w_t)$, hence, $\gamma(tt^{-1}) = \gamma(u_1 \cdots u_n u_n^{-1} \cdots u_1^{-1})$. It follows that $\gamma(tt^{-1}) \stackrel{*}{\Longrightarrow} {}_{s=f}\bullet\!\stackrel{u_1}{-}\!\bullet\!\stackrel{u_2}{-}\!\bullet\cdots\bullet\!\stackrel{u_n}{-}\!\bullet$. From here, an induction on n proves that $\gamma(tt^{-1}t) \stackrel{*}{\Longrightarrow} \gamma(t)$ and that $\gamma(tt^{-1}rr^{-1}) \stackrel{*}{\Longrightarrow} g \stackrel{*}{\Longleftarrow} \gamma(rr^{-1}tt^{-1})$, for some $g \in G_X$.

(2) is a proof by induction on the length of $\stackrel{*}{\Longleftrightarrow}$. So, it is enough to prove that $g \Longrightarrow g'$ implies $\mathrm{I} \vdash \tau(g) = \tau(g')$. Let g be as in (6) and suppose $\bullet_p\xleftarrow{a}\bullet_q\xrightarrow{a}\bullet_r$ is a subgraph of g (the other case is symmetric). Then, either it is a subgraph of T_j $(1 \leq j \leq n)$ or one of its edges is one of the u_i $(1 \leq i < n)$. Each of these cases has two subcases (the graphs on the left in the figure below show the four cases.) Below, "$\equiv$" indicates the use of definition of τ, and "$=$" the use of Axiom (4).
Case (A1). There is a subtree in T_j of the form $T = a^{-1}\{aT', T''\}$.

$$\begin{aligned}\tau(g) \equiv \cdots\tau(a^{-1}\{aT', T''\})\cdots &\equiv \cdots a^{-1}a\tau(T')a^{-1}\tau(T'')a\cdots \\ &\equiv \cdots\tau\{a^{-1}\{\bullet\}, T', a^{-1}T''\}\cdots\end{aligned}$$

$$\begin{aligned} &\equiv \cdots\tau\{a^{-1}\{\bullet\}, a^{-1}T'', T'\}\cdots \\ &\equiv \cdots a^{-1}aa^{-1}\tau(T'')a\tau(T')\cdots \\ &= \cdots a^{-1}\tau(T'')a\tau(T')\cdots \\ &\equiv \cdots\tau(\{a^{-1}T'', T'\})\cdots \equiv \tau(g'). \end{aligned}$$

Case (A2). There is a subtree in T_j of the form $T = \{aT', aT''\}$.

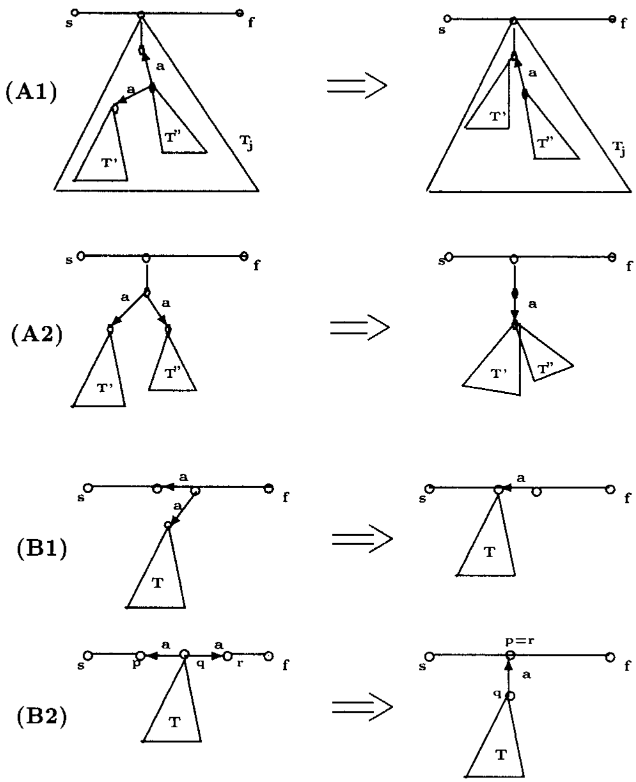
$$\begin{aligned} \tau(g) \equiv \cdots\tau(\{aT', aT''\}\cdots &\equiv \cdots a\tau(T')a^{-1}a\tau(T'')a^{-1}\cdots \\ &\equiv \cdots\tau(a\{T', a^{-1}\{\bullet\}, T''\})\cdots \\ &\equiv \cdots\tau(a\{a^{-1}\{\bullet\}, T', T''\})\cdots \\ &\equiv \cdots aa^{-1}a\tau(T')\tau(T'')a^{-1}\cdots \\ &= \cdots a\tau(T')\tau(T'')a^{-1}\cdots \\ &= \cdots\tau(a\{T, T''\})\cdots \equiv \tau(g'). \end{aligned}$$

Case (B1). There is i with $u_i = a^{-1}$ and $T_{i+1} = \{aT, \ldots\}$.

$$\begin{aligned} \tau(g) \equiv \cdots\tau(u_i)\tau(aT)\cdots &\equiv \cdots a^{-1}a\tau(T)a^{-1}\cdots \\ &\equiv \cdots\tau(\{a\{\bullet\}, T\})a^{-1}\cdots \\ &\equiv \cdots\tau(\{T, a^{-1}\{\bullet\}\})a^{-1}\cdots \\ &\equiv \cdots\tau(T)a^{-1}aa^{-1}\cdots \\ &= \cdots\tau(T)a^{-1}\cdots \equiv \tau(g'). \end{aligned}$$

Case (B2). There is i with $u_i = a^{-1}$ and $u_{i+1} = a$.

$$\begin{aligned} \tau(g) \equiv \cdots\tau(u_i)\tau(T)\tau(u_{i+1})\cdots &\equiv \cdots a^{-1}\tau(T)a\cdots \\ &\equiv \cdots\tau(a^{-1}T)\cdots \equiv \tau(g'). \end{aligned}$$



∎

Lemma 3. *1. There is no infinite sequence* $g_1 \Longrightarrow \cdots \Longrightarrow g_i \Longrightarrow \cdots$

2. If $h_1 \Longleftarrow g \Longrightarrow h_2$ *then there is* $\bar{g} \in G_X$ *such that* $h_1 \stackrel{*}{\Longrightarrow} \bar{g} \stackrel{*}{\Longleftarrow} h_2$.

Proof. For (1) just note that each $\Longrightarrow$-step diminishes by one the number of vertices of a finite graph. For (2), if the two subgraphs to be rewritten in g are edge-disjoints, then define $\bar{g}$ as the graph obtained from g by doing both rewritings (note that the order in which they are done does not matter). If the two subgraphs to be rewritten have common edges, g must have a subgraph like $\bullet \stackrel{a}{\longleftarrow} \bullet \stackrel{a}{\longrightarrow} \bullet \stackrel{a}{\longleftarrow} \bullet$. Observe that again $\bar{g}$ defined as before works. ∎

From the two statements of Lemma 3, a purely combinatorial argument shows that a "global" version of 3(2) also holds: *If* $h_1 \stackrel{*}{\Longleftarrow} g \stackrel{*}{\Longrightarrow} h_2$ *then there is* $\bar{g}$ *such that* $h_1 \stackrel{*}{\Longrightarrow} \bar{g} \stackrel{*}{\Longleftarrow} h_2$. In fact, something seemingly stronger, but actually equivalent to it, can be proved (for a discussion of these rewriting concepts and the missing proofs see [3]):

Lemma 4. *If* $g_1 \stackrel{*}{\Longleftrightarrow} g_2$ *then there is* g *such that* $h_1 \stackrel{*}{\Longrightarrow} g \stackrel{*}{\Longleftarrow} h_2$. ∎

Theorem 1. *Up to isomorphism, the free inverse semigroup on* X *consists of all isomorphism classes of birooted word-trees on* X.

Proof. Lemmas 3 and 4 show that each $\overset{*}{\Longleftrightarrow}$-class of graphs in G_X has a canonical representative: Consider any g in the class, and apply repeatedly $\Longrightarrow$ until it is no more applicable. By Lemma 3(1) this process stops, and from Lemma 4, the element obtained, denoted by $\mathrm{nf}(g)$, can be proved to be unique.

Define $B_X = \{\mathrm{nf}(g) : g \in G_X\}$. Observe that B_X is by definition the set of all birooted word-trees on X. Moreover it is a quotient-algebra of G_X. Denoting by $\langle I \rangle$ the congruence generated by the equations in I, and by FI_X the free inverse semigroup on X, we have:

$$FI_X \equiv T_X/\langle I \rangle \xrightarrow{\gamma} (G_X/\overset{*}{\Longleftrightarrow}) \xrightarrow{\mathrm{nf}} B_X.$$

Clearly nf is an isomorphism. Also γ is an isomorphism with inverse τ: Given $t \in T_X$, observe that $\tau\gamma(w_t) = w_t$, and using Lemma 2 it follows that $I \vdash \tau\gamma(t) = \tau\gamma(w_t) = w_t = t$. Similarly it can be shown that $\gamma\tau$ is the identity in G_X. Hence $FI_X \cong B_X$. ■

References

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[3] W. Wechler, "Universal Algebra for Computer Scientists", EATCS Monographs on TCS, Vol 25, Springer-Verlag, Berlin, New York, 1992.

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Received May 31, 1998
and in final form September 14, 1998